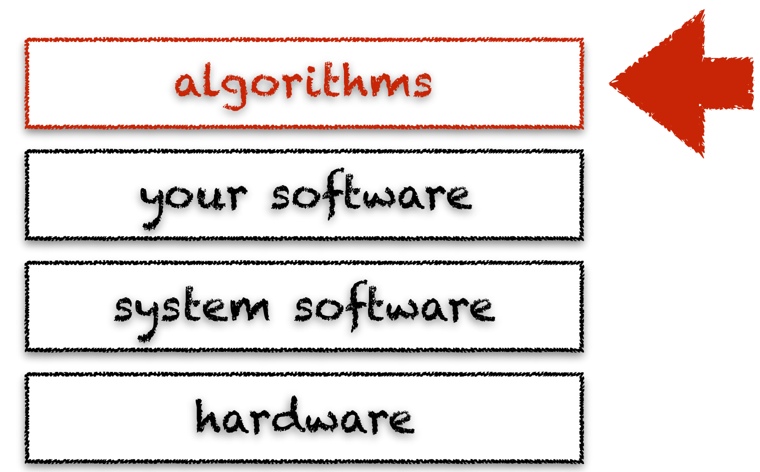
spatial tree algorithms

learning objectives



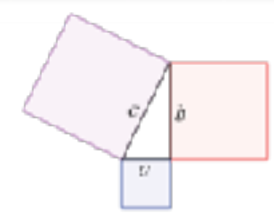
- learn the characteristics of spatial data
- learn several spatial indexing data structures
- learn basic algorithms for using such structures

computational geometry

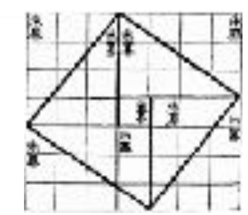
a branch of computer science focusing on data structures & algorithms for solving geometric problems

development made possible by exponential progress in computer graphics, with multiple applications

mathematical visualization, e.g., proof without words, mandelbrot sets

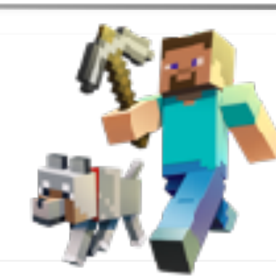


$a^2 + b^2 = c^2$



$z \mapsto z^d + c$

geographic information systems, e.g., location search & route planning



computer vision e.g., 3D graphics is games

computer-aided engineering, e.g., mechanical design

computational geometry

what's specific to spatial data?

with 1-dimensional data, natural ordering implicitly partitions the data, e.g., binary tree

spatial data is intrinsically multidimensional, so there is no natural ordering of data (e.g., of points)

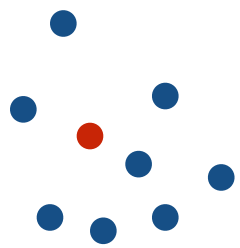
with 1-dimensional data, the static case is rather simple and solved by sorting the data

with multidimensional data, the static case is far from simple and solved by several partitioning techniques

computational geometry

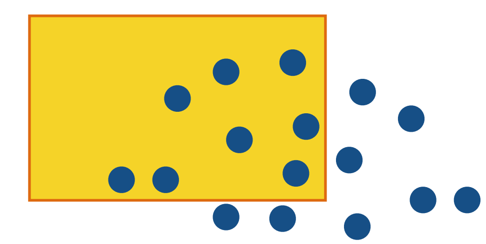
typical problems

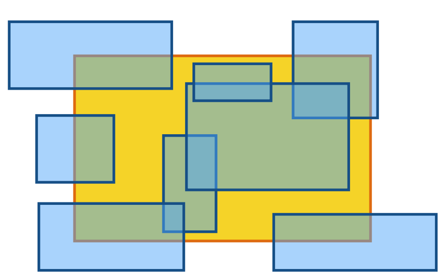
y



nearest neighbor: given a set of points P, find which one is closest to a target point p_t

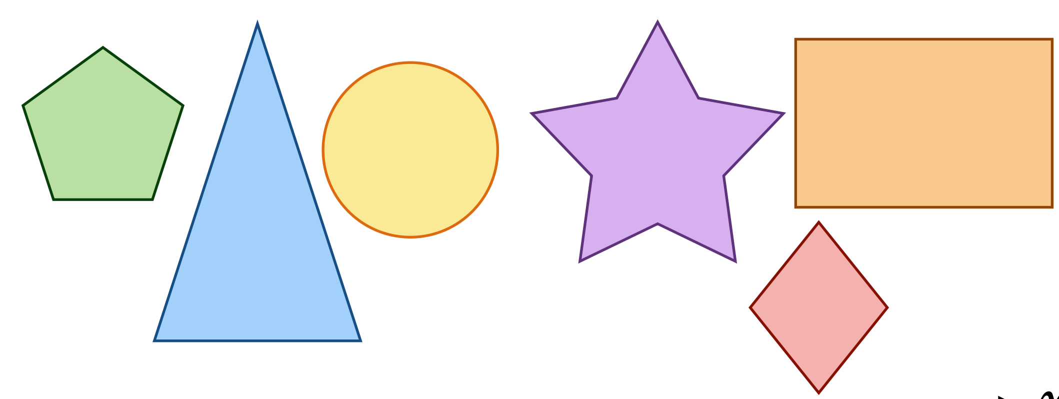
range queries: given a set of points P, find the points contained within a given rectangle





intersection queries: given a set of rectangles R, find which rectangles intersect a target rectangle

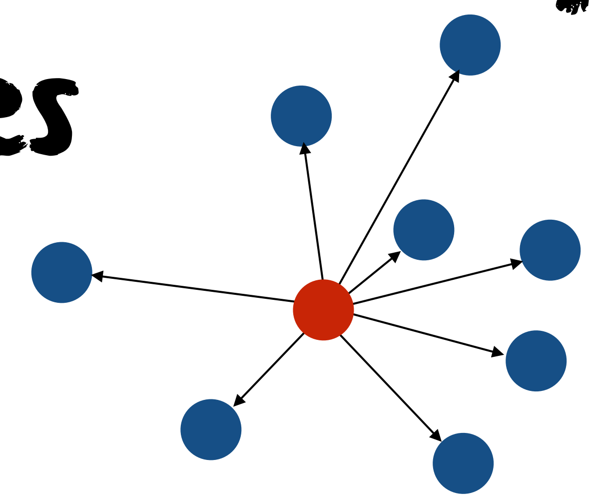
collision detection: given a set of shapes S, find the intersections between all these shapes



x

computational geometry

typical approaches



brute-force algorithm

nearest neighbor: given a set of points P, find which one is closest to a target point p_t

Complexity: $O(n)$, with $n = |P|$

NEAREST–NEIGHBOR (P, p_t)
  $p \leftarrow$ NIL
  $min \leftarrow \infty$
  for each $p_i \in P$
    if $distance(p_i, p_t) < min$
      $min \leftarrow distance(p_i, p_t)$
      $p \leftarrow p_i$
  return (p, min)

spatial tree structures

they index spatial objects

quad-trees

kd-trees

R-trees

Complexity: $O(\log n)$, with $n = |P|$

R-tree

A. Guttman. ***R-trees: A dynamic index structure for spatial searching***. In Proceedings of the 1984 ACM SIGMOD International Conference on Management of Data, pages 47–57, New York, NY, USA, 1984. ACM.

a recursive tree, where each node has between M and $m = \left\lfloor \frac{M}{2} \right\rfloor$ children, except for the root which has at least two

only leaf nodes contain actual spatial object entries, each consisting of the spatial object itself and a minimum bounding region (mbr) containing that object, i.e., $object = (shape, mbr)$

internal nodes contain children entries, each consisting of a link to the child node and an mbr covering all children nodes of that child, i.e., $node = (child, mbr)$

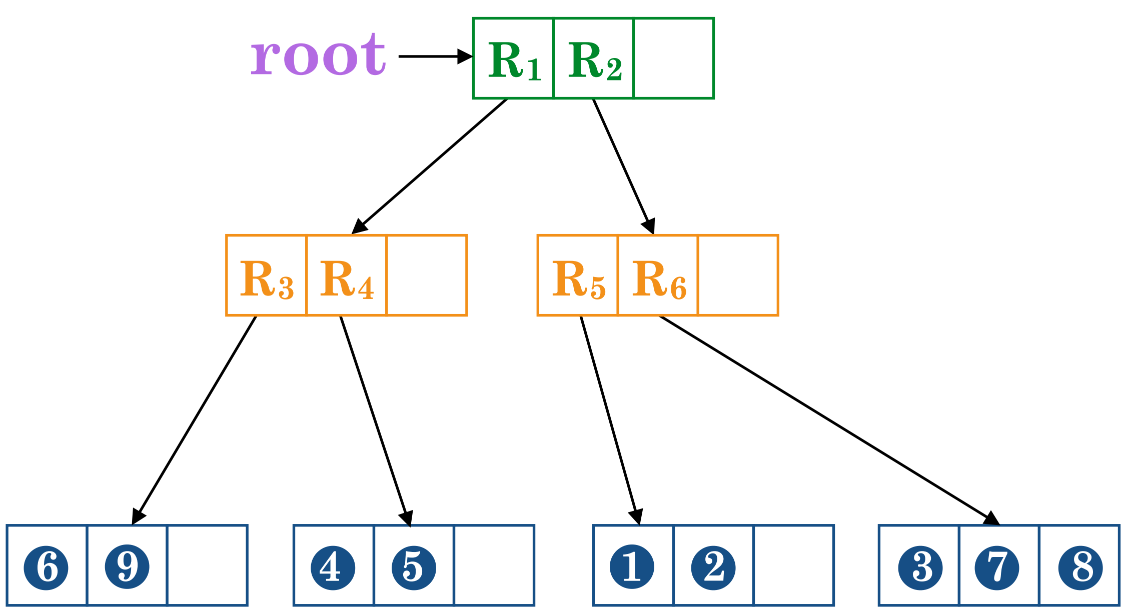
an minimum bounding region is typically of the form $mbr = (x_{min}, y_{min}, x_{max}, y_{max})$

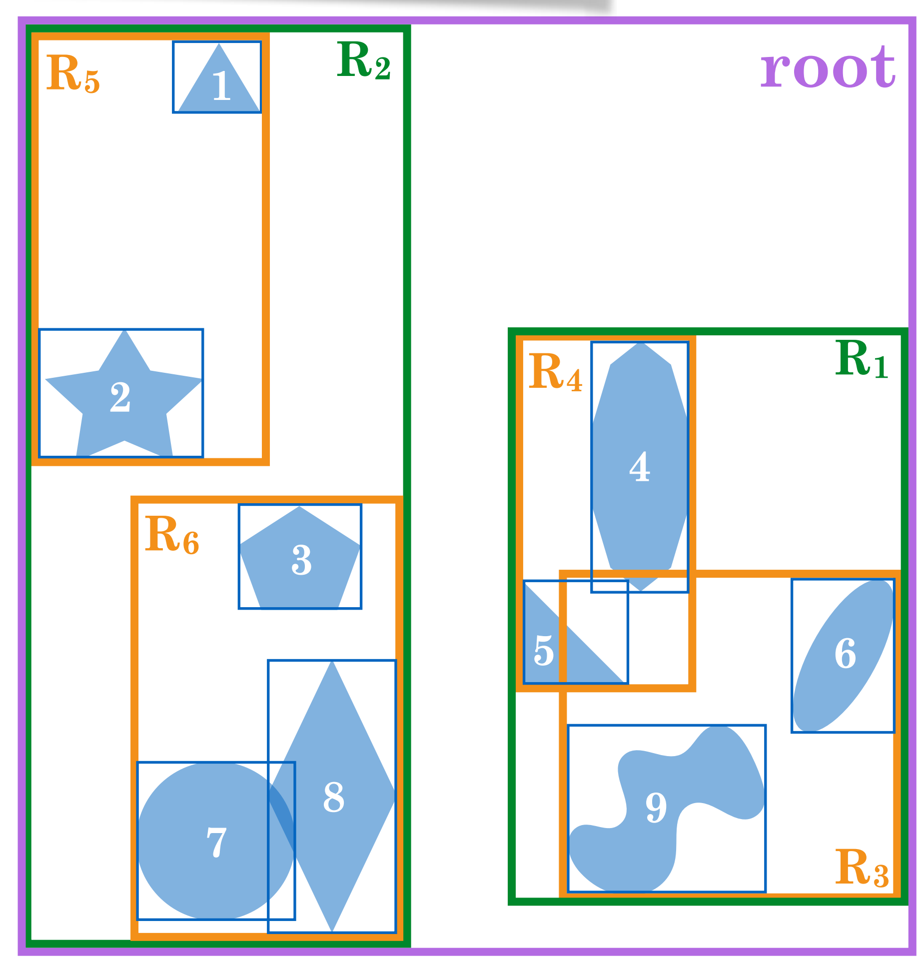
all leaves are at the same level, i.e., the tree is height balanced

R-tree

only leaf nodes contain actual spatial object entries, each consisting of the spatial object itself and a **minimum bounding region (mbr)** containing that object, i.e., $object = (shape, mbr)$

internal nodes contain children entries, each consisting of a link to the child node and an **mbr covering all children nodes** of that child, i.e., $node = (child, mbr)$

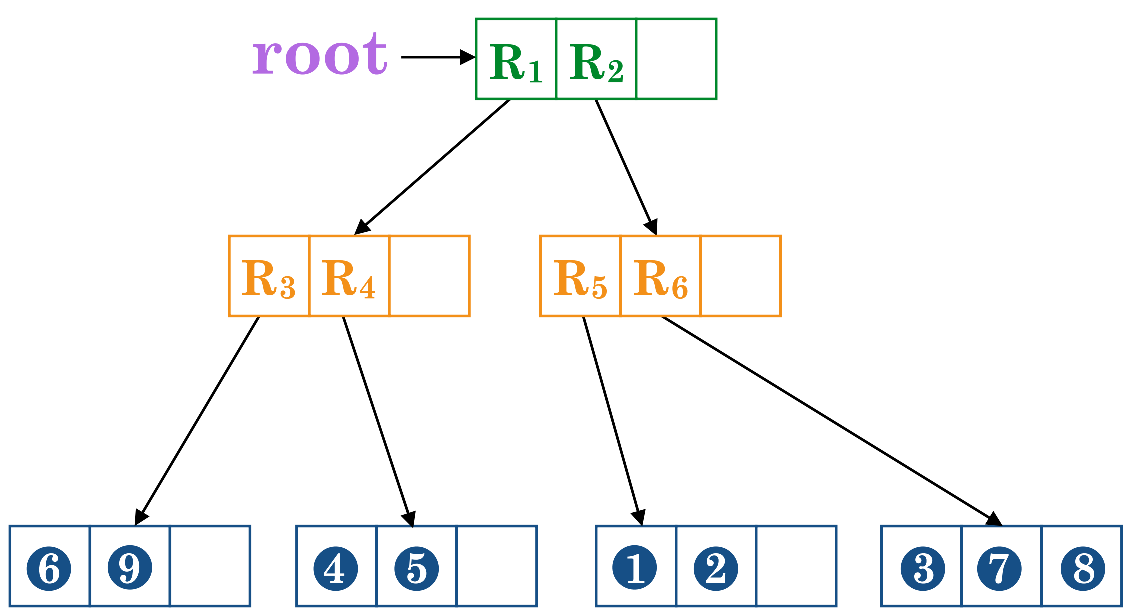




important: the root also contains a minimum bounding box

R-tree

INTERSECT (*node, rectangle*)
  if $node.mbr \subset rectangle$
    return $\{\, object \mid object \in$ REACHABLE-LEAVES($node$) $\}$
  else if *node is a leaf*
    return $\{\, object \in node \mid object.mbr \cap rectangle \neq \varnothing \,\}$
  $result \longleftarrow \varnothing$
  for each $child \in node.children$
    if $child.mbr \cap \text{rectangle} \neq \varnothing$
      $result = result \cup$ INTERSECT ($child$, rectangle)
  return *result*



important: the root also contains a minimum bounding box

SEARCH (*node, shape*)
  if *node is a leaf*
    if $\exists\, object \in node : object.shape = shape$
      return *object*
    else return NIL
  for each $child \in node.children$
    if $shape.mbr \subseteq child.mbr$
      return SEARCH($child$, $shape$)
  return NIL

quad-tree

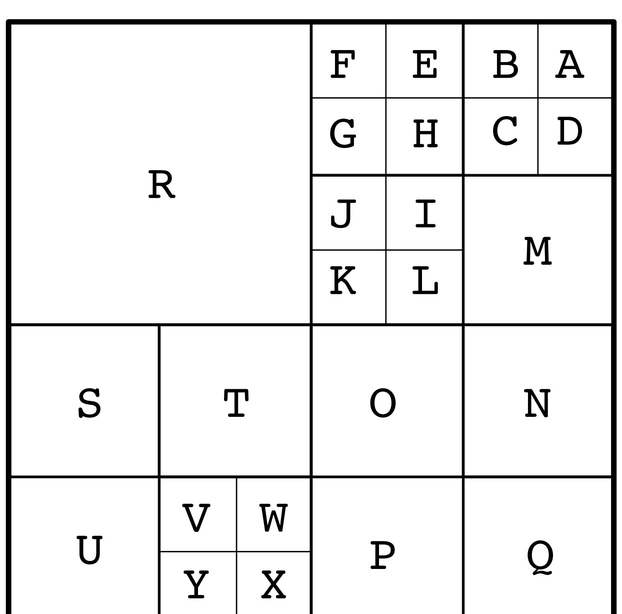
R. A. Finkel and J. L. Bentley. ***Quad trees a data structure for retrieval on composite keys.*** Acta Informatica, 4(1):1–9, 1974.

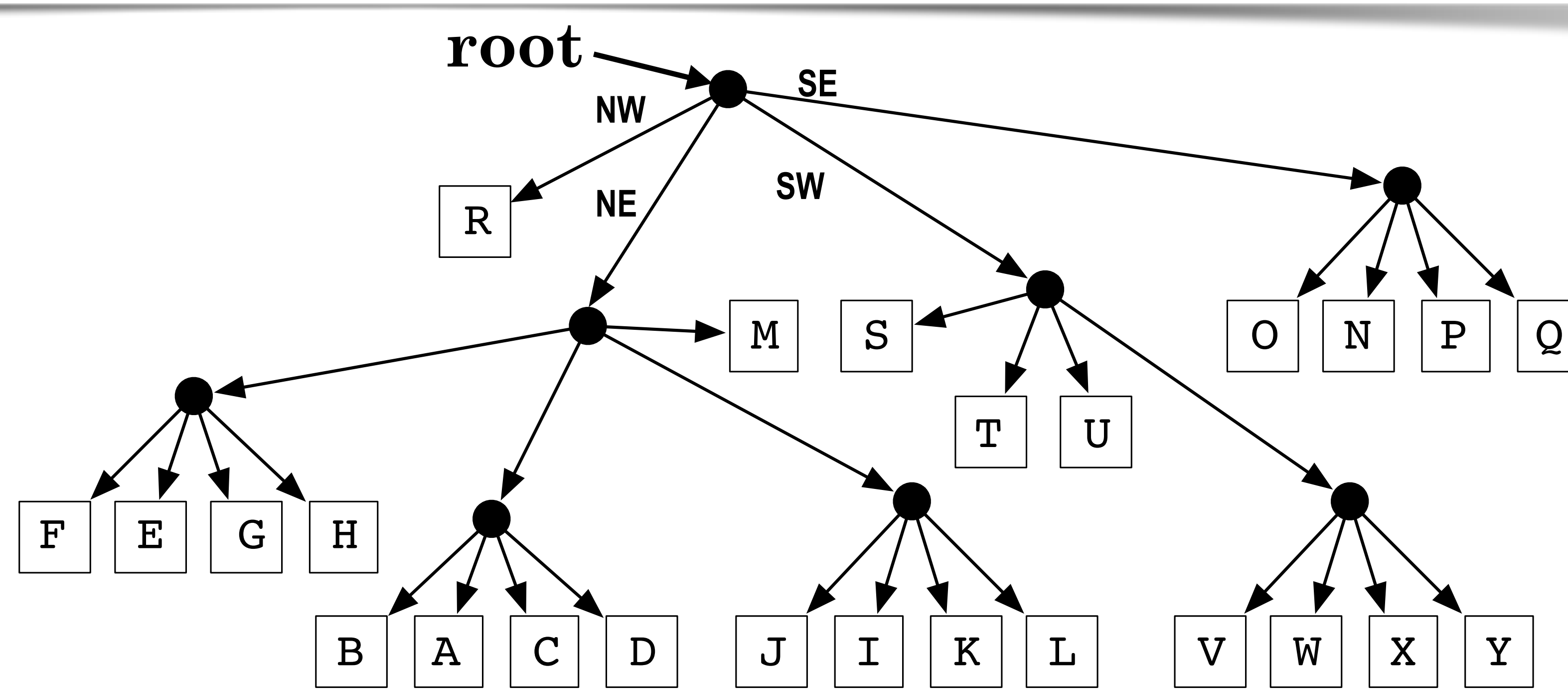
a recursive tree where each internal node has four children

each node represents a cell in the geometrical space, with its children partitioning that cell into an equally sized subcell

predefined partitioning with subcells (quadrants) named as North West (NW), North-East (NE), South-West (SW) and South-East (SE)

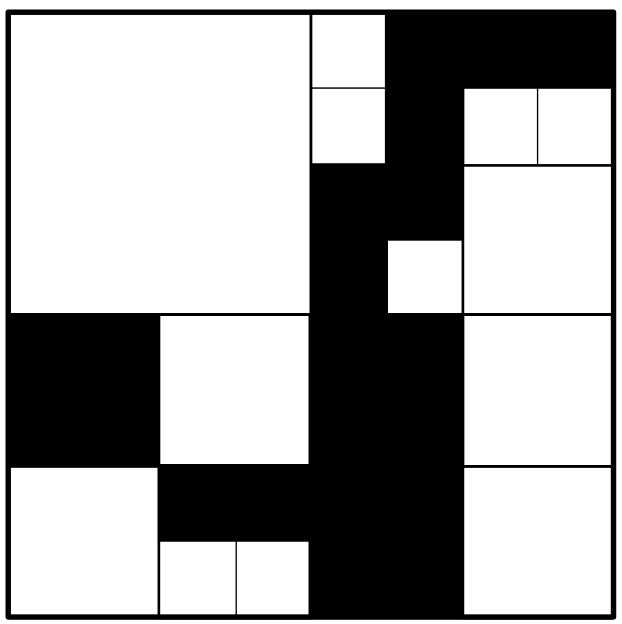
like R-trees, only leaf nodes store actual geometrical objects

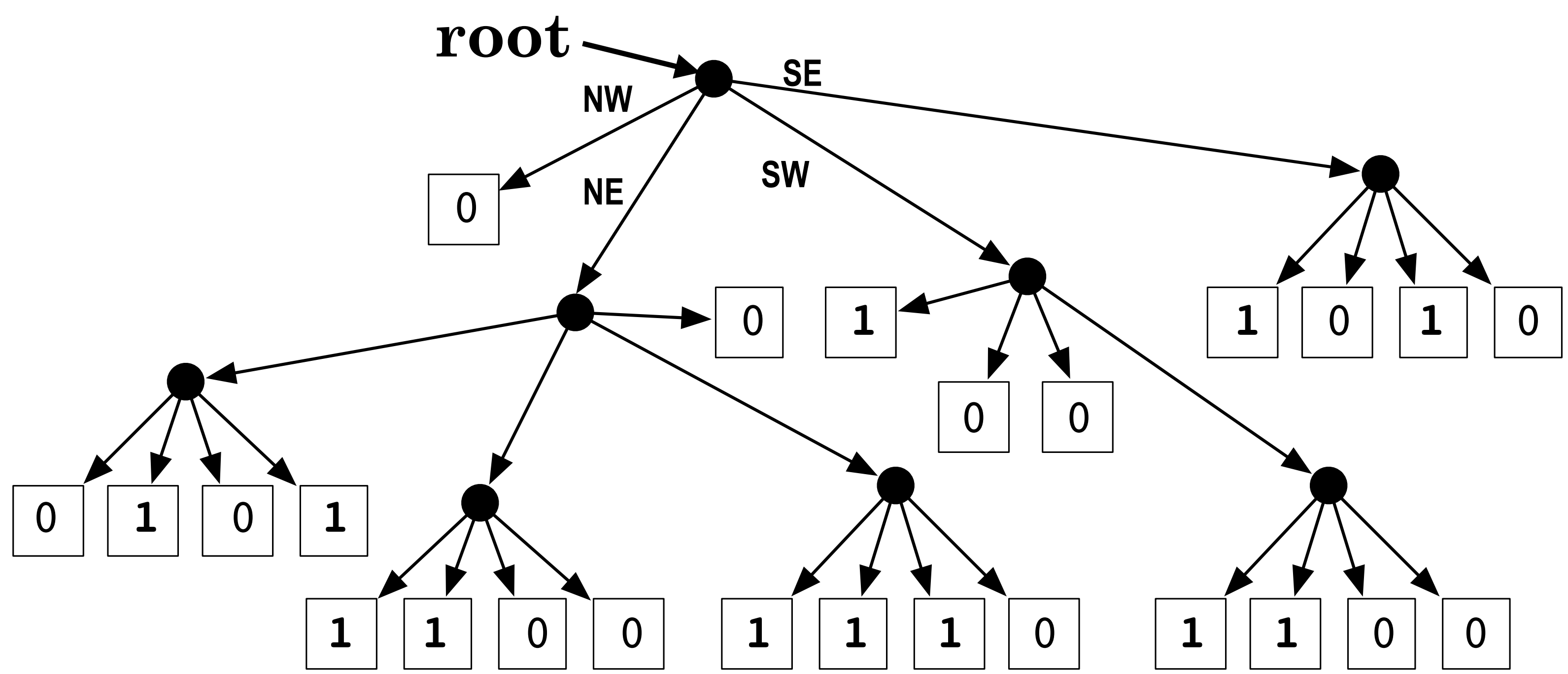




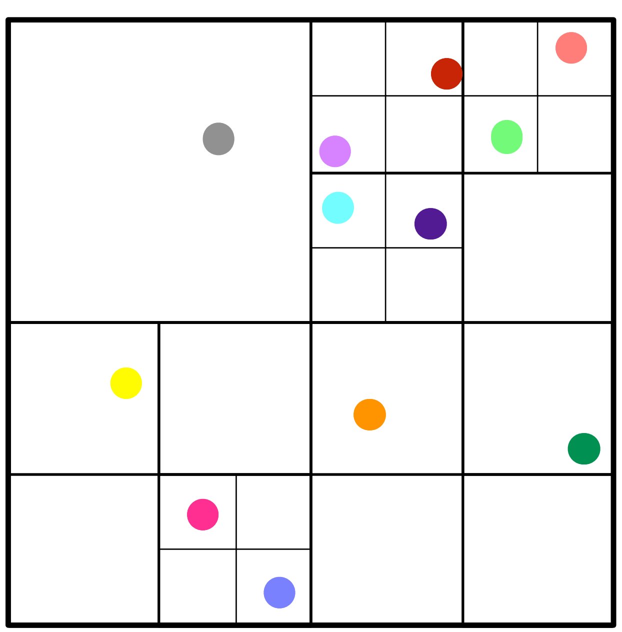
quad-tree

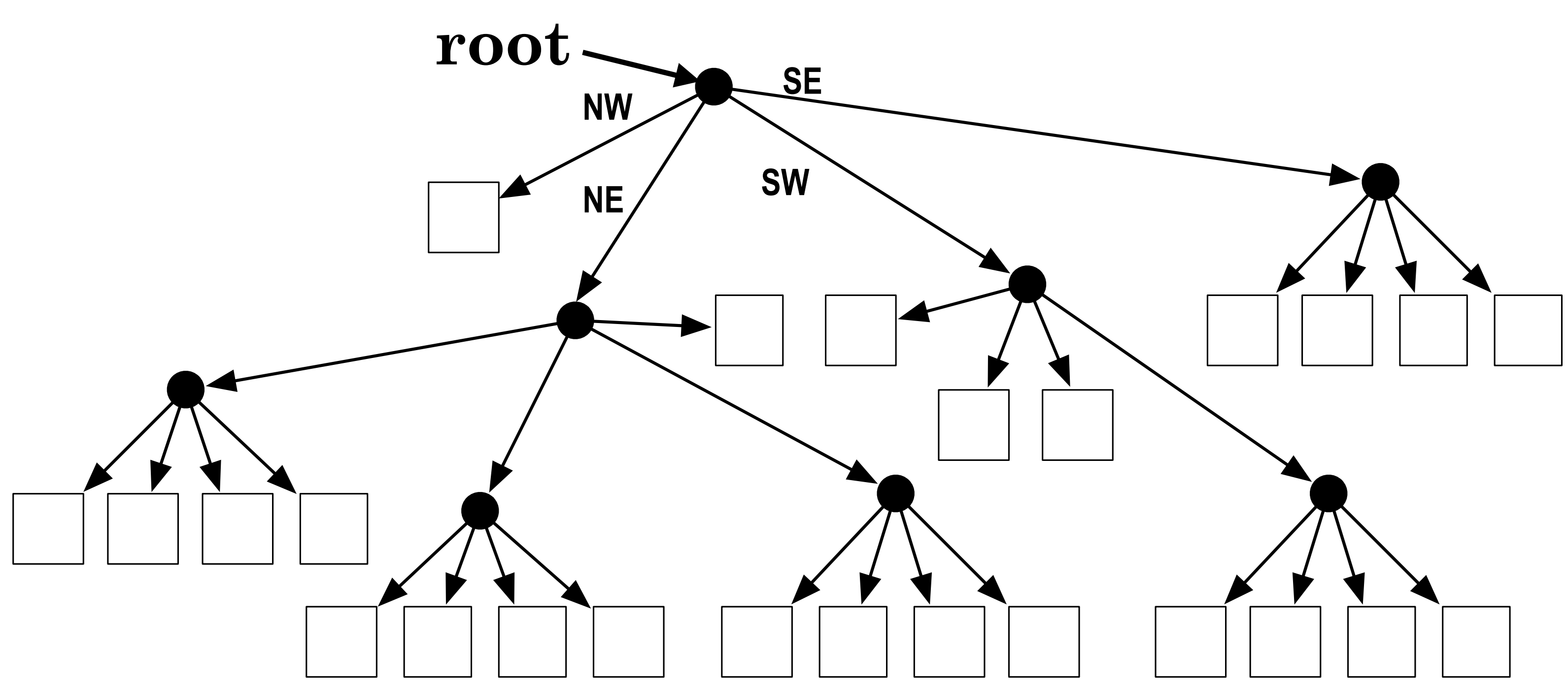
region quad-tree





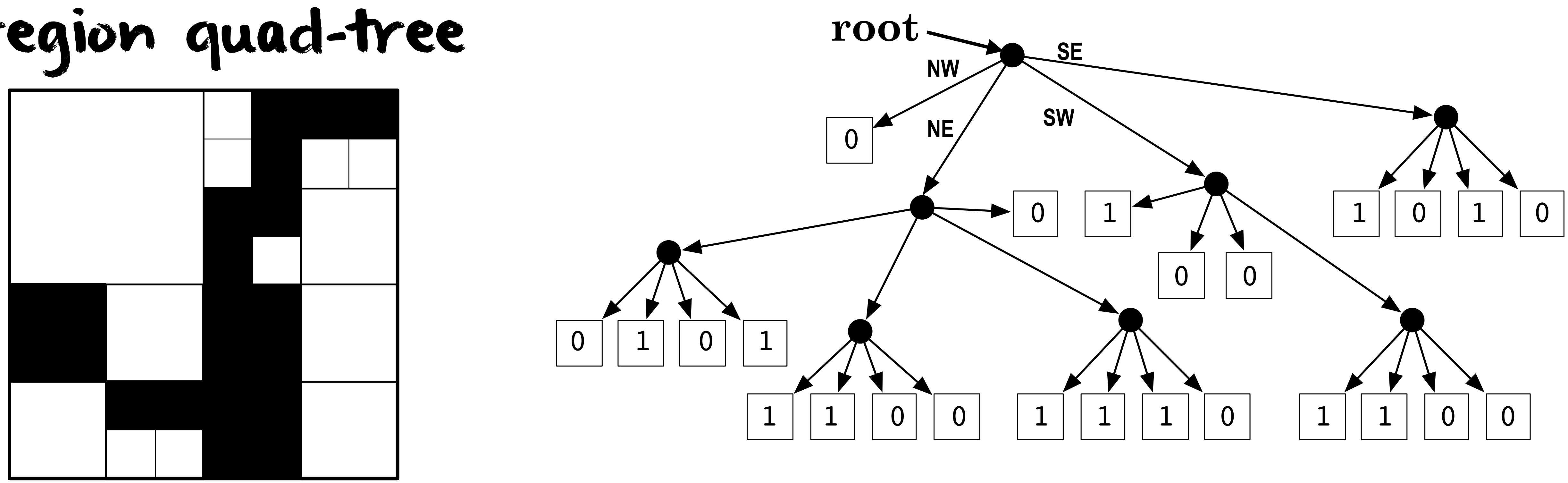
point-region quad-tree



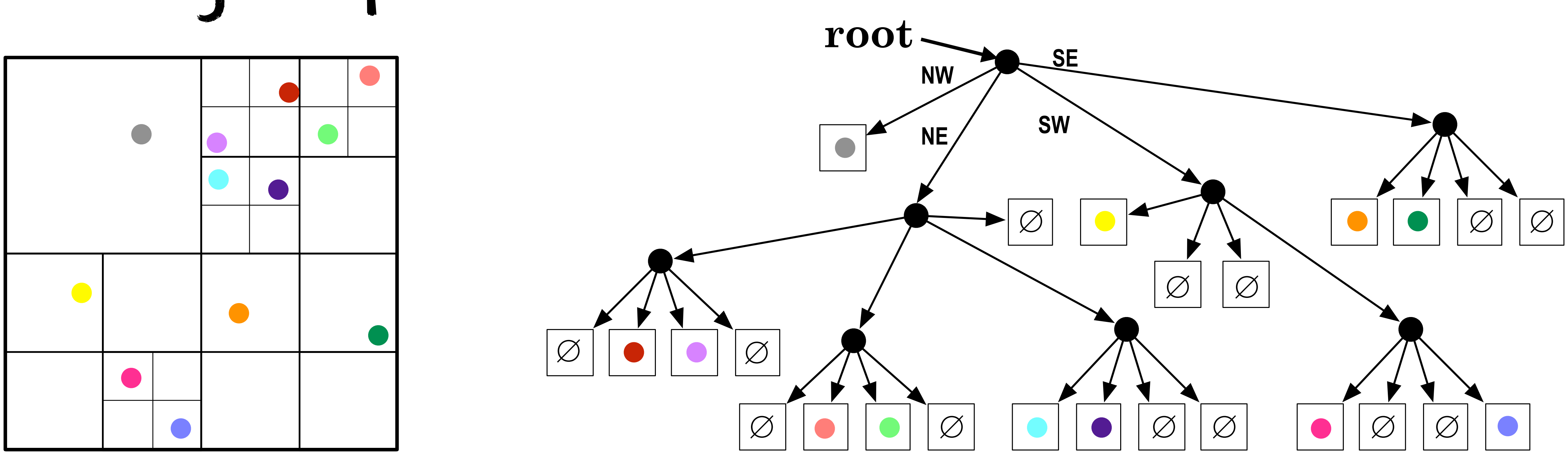


quad-tree

region quad-tree



point-region quad-tree

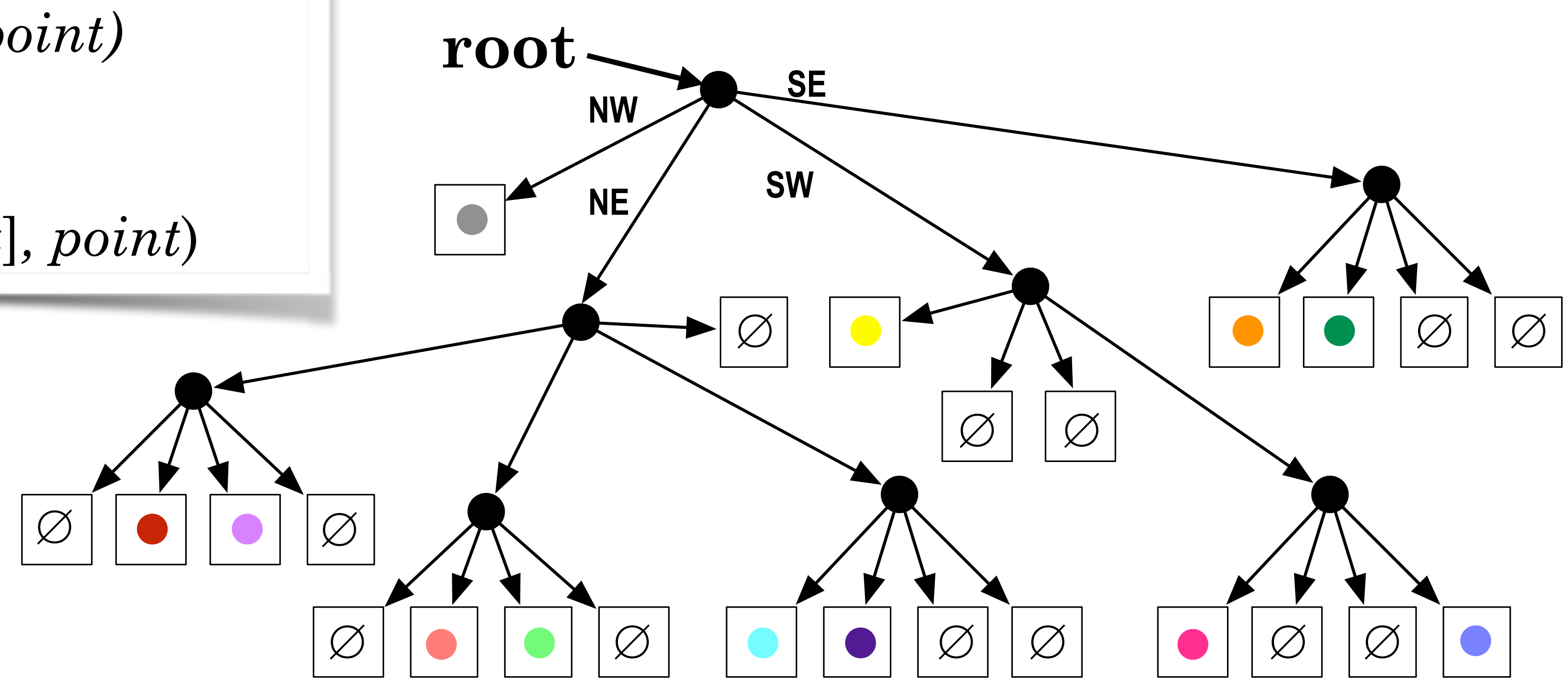


quad-tree

```
INTERSECT (node, rectangle)
    if node is a leaf
        if node.point ⊂ rectangle return { node.point }
        else return ∅
    else if node.cell ⊂ rectangle
        return { node.point | node ∈ REACHABLE-LEAVES(node) }
    else
        result ⟵ ∅
        for each child ∈ node.children
            if child.cell ∩ rectangle ≠ ∅
                result = result ∪ INTERSECT (child, rectangle)
    return result
```

```
ADD (node, point)
    if point ⊂ node.cell
        return FALSE
    if node is a leaf
        if node.point = point
            return FALSE
        if node.point = NIL
            node.point ⟵ point
            return TRUE

    quadrant ⟵ FIND-QUADRANT(node, point)
    if node is a leaf
        SUBDIVIDE(node)
    return ADD (node.children[quadrant], point)
```



kd-tree

J.L. Bentley. ***Multidimensional binary search trees used for associative searching***. Commun. ACM, 18(9):509–517, September 1975.

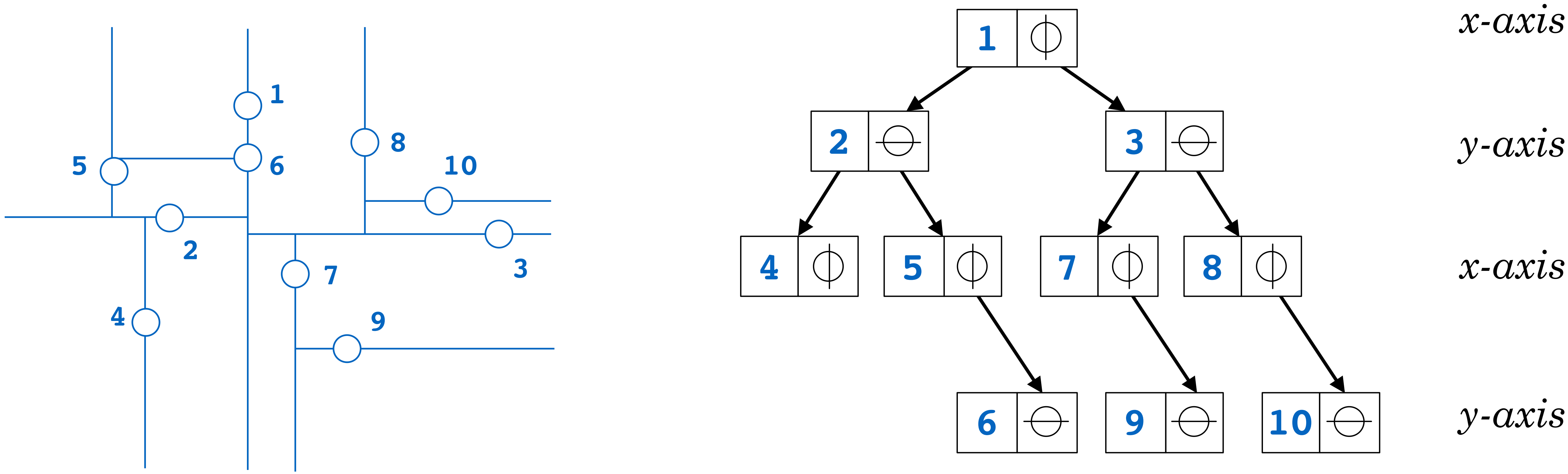
a kd-tree (short for k-dimensional tree) is a binary tree in which every node is a k-dimensional point

in addition, each internal node divides the k-dimensional space into two parts known as half-spaces

all points in one half space are contained in the left subtree of the node and all points in the other half space contained in the right subtree

all nodes at the same level (height) divide the k-dimensional space according to the same cutting dimension (axis)

k-d-tree



Add (*node*, *point*, *cutaxis*)
  if *node* = NIL
    node $\leftarrow$ Create-Node
    node.point = *point*
    return *node*
  if *point*[*cutaxis*] $\leq$ *node.point*[*cutaxis*]
    node.left = Add(*node.left*, *point*, *cutaxis* **mod** *k*)
  else
    node.right = Add(*node.right*, *point*, *cutaxis* **mod** *k*)
  return *node*